

# Topology Induced Spatial Bose-Einstein Condensation for Bosons on Star-Shaped Optical Networks

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(February 2, 2008)

## Abstract

New coherent states may be induced by pertinently engineering the topology of a network. As an example, we consider the properties of non-interacting bosons on a star network, which may be realized with a dilute atomic gas in a star-shaped deep optical lattice. The ground state is localized around the star center and it is macroscopically occupied below the Bose-Einstein condensation temperature  $T_c$ . We show that  $T_c$  depends only on the number of the star arms and on the Josephson energy of the bosonic Josephson junctions and that the non-condensate fraction is simply given by the reduced temperature  $T/T_c$ .

## I. INTRODUCTION

Although it is well known that free bosons hopping on translationally invariant networks cannot undergo Bose-Einstein condensation at finite temperature if the space dimension  $d$  is less or equal to two (see Ref. [1]), very recent studies [2,3] hint to the exciting possibility that the network topology may act as a catalyst for inducing a finite temperature spatial Bose-

Einstein condensation even if  $d < 2$ . As an example of this situation we shall investigate the properties of non-interacting bosons hopping on a star shaped optical network, evidencing that - already for this very simple graph topology - one may have a macroscopic occupation of the ground-state at low temperatures.

A star graph (see Fig. 1) is made of  $p$  one-dimensional chains (*arms*) which merge in one point called the center of the star. Each site  $i$  of the star arms is naturally labeled by two integer indices  $x$  and  $y$  where  $x = 0, \dots, L$  labels the distance from the center and  $y = 1, \dots, p$  labels the arms. The center of the network is denoted by  $O \equiv (0, y)$ . The total number of sites is  $N_s = (pL + 1)$ ; each site on the arm is linked only to two neighbors whereas the center has  $p$  neighbors: thus the fact that the center has coordination number  $p$  is the source of spatial inhomogeneity in this lattice. In the following we shall evidence that bosons hopping on this graph undergo - at a certain temperature  $T_c$ , which depends on the number of star arms - a topology induced spatial Bose-Einstein condensation in a state localized around the center of the star graph.

Bosons hopping on star-shaped networks can be experimentally realized loading a dilute Bose-Einstein condensate (BEC) in a suitable periodic potential, arranged to provide a star-like configuration. Periodic potentials are, nowadays, routinely created with two or more counterpropagating laser beams; one can accurately tune the height of the potential (which is proportional to the power of the lasers) as well as the distance between neighboring sites. When one loads an atomic BEC in a deep optical lattice, one has a bosonic Josephson network, i.e., an array of bosonic Josephson junctions (BJJ's). A single BJJ may be obtained by loading a BEC in a double well potential: the weak link between the atomic condensates is provided by the energy barrier between them and the dynamics of the atoms at  $T = 0$  is described by Josephson equations, obtained from the Gross-Pitaevskii equation [4]. Similarly, with a multi-well periodic potential, when the heights of the barriers are much higher than the chemical potential, the system realizes a lattice of weakly coupled condensates. Each bosonic Josephson junction consists of a pair of neighboring condensates: the tunneling rate (proportional to the Josephson energy) is easily tuned by changing the

power of the lasers, and it decreases if the interwell barriers increase. The properties of a bosonic Josephson network at  $T = 0$  are described by a discrete nonlinear Schrödinger equation, obtained from the Gross-Pitaevskii equation with a periodic potential [5]. We remark that recently it has been showed that a  $2D$  optical network of BEC can be described by the Gross-Pitaevskii equation also at finite temperature, provided that the interwell energy barriers and the frequency of the axial confinement are large enough [6].

Present-day BJJ networks are built on regular geometries: linear chains [7], squares [8] and cubes [9]. However, a variety of non-conventional structures may be produced by the standing waves of several interfering laser beams suitably placed [10]. For instance, a star with four arms may be realized by having two perpendicular gaussian laser beams superimposed to a  $2D$  optical lattice; the center of the star should be then arranged by adjusting the distance between neighboring wells.

BJJ networks may be pertinently described by the Bose-Hubbard Hamiltonian [11,12]. In fact, when all the relevant physical parameters are small compared to the excitation energies, the field operator [12], describing the condensate configuration in the BJJ network, may be expanded as  $\hat{\psi}(\vec{r}, \tau) = \sum_j \hat{a}_j(\tau) \phi_j(\vec{r})$  with  $\phi_j(\vec{r})$  the normalized Wannier wavefunction localized in the  $j$ -th well and  $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$  the bosonic number operator. Substituting the expansion for  $\hat{\psi}(\vec{r}, \tau)$  in the full quantum Hamiltonian describing the bosonic system [12] leads to a Bose-Hubbard model

$$H = -t \sum_{\langle i,j \rangle} (\hat{a}_i^\dagger \hat{a}_j + h.c.) + \frac{U}{2} \sum_j \hat{n}_j (\hat{n}_j - 1). \quad (1)$$

In Eq. (1)  $\sum_{\langle i,j \rangle}$  denotes a sum over all the distinct pairs of neighboring sites,  $\hat{a}_j^\dagger$  ( $\hat{a}_j$ ) is the bosonic operator which creates (destroys) a boson at site  $j$ ,  $U = (4\pi\hbar^2 a/m) \int d\vec{r} \phi_j^4$  ( $a$  is the s-wave scattering length and  $m$  is the atomic mass) and  $t \simeq - \int d\vec{r} [\frac{\hbar^2}{2m} \vec{\nabla} \phi_i \cdot \vec{\nabla} \phi_j + \phi_i V_{ext} \phi_j]$ , where  $\phi_i$  and  $\phi_j$  are the Wannier functions at the neighboring sites  $i$  and  $j$  and  $V_{ext}$  is the external potential confining the bosons. Since, for atomic condensates,  $U$  may be varied by tuning the scattering length using Feshbach resonances [1], one may assume that  $U \ll t$ ; for the sake of simplicity, in the following we shall set  $U = 0$ .

The filling, i.e., the average number of particles per site, is defined as  $f = N_T/N_s$ , where  $N_T$  is the total number of bosons. When  $f \gg 1$  and the fluctuations of the particle numbers per site are much smaller than  $f$ , one can safely substitute the operator  $\hat{a}_i$  with  $\sqrt{N_i} e^{i\phi_i}$ , where  $N_i$  is the number of particles at the site  $i$  [14,15]. As a result, the Josephson energy of a single BJJ is given by

$$E_J \approx 2tf. \quad (2)$$

It is worthwhile to remark that the Bose-Hubbard Hamiltonian (1) describes not only BEC's, but in general cold bosons in a deep periodic potential, provided that the temperature does not excite higher bands:  $E_{gap} \gtrsim k_B T$ , where  $E_{gap} \propto \sqrt{V_0}$  is the band energy gap and  $V_0$  is the interwell energy barrier. In this paper we are considering a BEC in a deep optical lattice: thus we are also assuming that in each well one has a grain condensate, i.e., that one works at a temperature smaller than the temperature  $T_{BEC}$  at which condensation occurs in a single well. For  $T < T_{BEC}$ , one has a network of weakly coupled condensates, and the bosons, at the temperature  $T_c$  defined by Eq. (14), start to macroscopically occupy the ground-state whose wavefunction is given by Eq. (8). We observe that for  $V_0 \sim 15E_R$ , where  $E_R = \hbar^2/2m\lambda^2$  with  $\lambda$  the wavevector of the lattice beams, and for an average number of particles per site  $f \sim 200$ , one has  $T_{BEC} \sim 500nK$  and  $T_c \sim 50nK$  [6]. The experimental situation in which the total number of particles is so small that  $T_{BEC}$  is much lower than  $T_c$  should be also possible: in this case for  $T > T_c$  one has not an array of BJJ's, but at  $T_c$  the bosons can still macroscopically occupy the ground-state described by the eigenfunction (8).

The plan of the paper is the following. In Sec. II we analyze the spectrum of bosons hopping on a star-shaped graph; we shall evidence that, due to the network inhomogeneity, an isolated localized ground-state appears in the spectrum (its effect is similar to the one induced by an impurity on a linear chain, see e.g. Ref. [13]); we shall also determine, as a function of the number of star arms  $p$ , the energy gap between the ground-state and the continuum states as well as the coherence length  $\xi$  which characterizes the topology

induced spatial condensation of bosons in the center of the star graph. In Sec. III we analyze the thermodynamic properties of non-interacting bosons hopping on the star graph and we determine the critical temperature  $T_c$  as a function of  $p$  and of the Josephson energy of the single BJJ; we show that the condensate fraction and the inhomogeneous spatial distribution of bosons over the array can be expressed in a simple way as a function of the scaled temperature  $T/T_c$ . Section IV is devoted to some remarks on the results of our investigation and to a discussion of experimental settings which should enable to evidence the existence of the topology induced spatial BEC analyzed in this paper.

## II. SPECTRUM OF BOSONS HOPPING ON A STAR GRAPH

The topology and geometry of a generic graph network is fully described by its adjacency matrix  $A_{x,y;x',y'}$  whose entries are 1 if  $(x, y; x', y')$  is an allowed link and 0 otherwise; for a graph network, the Hamiltonian (1) is written as

$$H = -t \sum_{x,y;x',y'} A_{x,y;x',y'} \hat{a}_{x,y}^\dagger \hat{a}_{x',y'}. \quad (3)$$

The single-particle energy spectrum on a star network is found then by solving the eigenvalue equation [3]:

$$-t \sum_{x',y'} A_{x,y;x',y'} \psi_E(x', y') = E \psi_E(x, y), \quad (4)$$

with the adjacency matrix  $A_{x,y;x',y'}$  given by:

$$A_{x,y;x',y'} = (\delta_{x',x-1} + \delta_{x',x+1}) (1 - \delta_{x,0}) \delta_{y,y'} + \delta_{x,0} \delta_{x',1}. \quad (5)$$

We shall refer the interested reader to the Appendix for the mathematical details of the solution of the eigenvalue equation (4) with adjacency matrix (5); here, we only describe the properties of the spectrum which turn out to be relevant for our subsequent analysis.

The spectrum  $\sigma$  is formed by  $N_s$  states and is divided in three parts:  $E_0$ ,  $\sigma_0$  and  $E_+$ .  $\sigma_0$  describes the delocalized states with energies between  $-2t$  and  $2t$ ; in the thermodynamic limit  $L \rightarrow \infty$ , the normalized density of these states is given by

$$\rho(E) = \frac{1}{\pi \sqrt{4t^2 - E^2}}, \quad (6)$$

just as for a particle hopping on a linear chain. Apart from the  $(pL - 1)$  continuum states belonging to  $\sigma_0$ , there are two bound states confined away from the continuum corresponding to energies  $E_0 < -2t$  and  $E_+ > 2t$ . These two eigenstates are localized and form the so-called *hidden* spectrum [2,3,16]: hidden means here that the two states - in the thermodynamic limit - do not contribute to the normalized density of states yielding the closure relation.

$E_0$  is the ground-state energy which, in the thermodynamic limit and for a star graph with  $p$  arms, is given by

$$E_0 = -t \frac{p}{\sqrt{p-1}}. \quad (7)$$

Equation (7) reproduces exactly the known result  $E_0 = -2t$  for  $p = 2$  and it implies the well known fact that, for a linear chain, there are no localized states. The energy of the isolated eigenstate in the high-energy region is simply given by  $E_+ = |E_0|$ .

It is worth observing that the spectrum is gapped: in fact, there is a finite gap  $\Delta = |E_0| - 2t$  between the ground-state energy and the continuum part of the spectrum; the value of  $\Delta$  depends on the number of arms  $p$ . In Fig. 2 we plot the energy gap as a function of the number of arms. As expected, one has  $\Delta = 0$  when  $p = 2$ .

The ground-state and the eigenstate corresponding to the eigenvalue  $E_+$  - due to the topology of the array - are localized in the center and exhibit an exponential decay in the direction of the arms. In the thermodynamic limit, the normalized ground-state wavefunction, as a function of the number of the arms  $p$ , is given by:

$$\psi_{E_0}(x, y) = \sqrt{\frac{p-2}{2p-2}} e^{-x/\xi} \quad (8)$$

(the normalization is, of course, over the whole network: i.e.,  $\sum_{x,y} |\psi_{E_0}(x, y)|^2 = 1$ ).  $\xi$  provides an estimate of the ground-state localization and it is given by

$$\xi = \frac{2}{\log(p-1)}. \quad (9)$$

In Fig. 3 we plot the ground-state wavefunction for different values of the number of arms  $p$ . Figure 3 evidences that adding arms enhances the localization of the wavefunction around the center of the star graph. As we shall see in the next section, bosons are allowed to spatially condense in this ground-state at low temperatures.

### III. THERMODYNAMICS OF BOSONS HOPPING ON A STAR GRAPH

The thermodynamical properties of non-interacting bosons hopping on a star-graph hint to the possibility of a topology induced spatial BEC in the center of the star graph [3]. To elucidate this phenomenon, it is most convenient to introduce the macrocanonical ensemble to determine the fugacity  $z$  as a function of the temperature of the system. The equation determining  $z$  is given by

$$N_T = \sum_{E \in \sigma} \frac{d(E)}{z^{-1} e^{\beta(E-E_0)} - 1}. \quad (10)$$

In Eq. (10)  $d(E)$  is the degeneracy of each single-particle eigenstate,  $E_0$  is the energy of the ground-state of the Hamiltonian (3) and  $\beta = 1/k_B T$ ; the sum in Eq. (10) is over the entire spectrum  $\sigma$ . For free bosons hopping on a star graph, one has

$$N_T = N_{E_0} + N_{E_+} + \int_{E \in \sigma_0} dE \frac{N_s \rho(E)}{z^{-1} e^{\beta(E-E_0)} - 1}, \quad (11)$$

where  $N_{E_0}(p, L; T)$  and  $N_{E_+}(p, L; T)$  denote, respectively, the number of bosons which occupy the ground-state and the state of energy  $E_+$  at a certain temperature  $T$ .  $\rho(E)$ , with  $E \in \sigma_0$ , is the density of delocalized states defined in Eq. (6). It is pertinent to define also the number of particles per site in each part of the spectrum as  $n_{E_0} = N_{E_0}/N_s$ ,  $n_{\sigma_0} = \int_{E \in \sigma_0} dE \rho(E) [z^{-1} e^{\beta(E-E_0)} - 1]^{-1}$  and  $n_{E_+} = N_{E_+}/N_s$ . In the thermodynamic limit, one has

$$n_{E_0}(T) = \lim_{L \rightarrow \infty} \frac{1}{N_s} \frac{1}{z^{-1} - 1},$$

and  $n_{E_+} = 0$  since

$$n_{E_+}(T) = \lim_{L \rightarrow \infty} \frac{1}{N_s} \frac{1}{z^{-1}e^{-2\beta E_0} - 1} < \frac{1}{e^{-2\beta E_0} - 1} = 0 \quad \forall T.$$

Thus, in the thermodynamic limit,  $E_+$  is not macroscopically occupied at any temperature and does not play any role in describing the thermodynamics of the system.

The last term of the right-hand side of Eq. (11) represents the number of bosons in the delocalized (chain-like) states. The presence of the hidden spectrum changes the behavior of the integral evaluated in the interval  $\{-2t, 2t\}$ , since it reduces it to the one describing non-interacting bosons on a linear chain with an impurity in one of the sites. As a result, letting  $z \rightarrow 1$ , the integral converges, even at finite temperatures, making possible the topology induced spatial BEC in the center of the star graph.

### A. Critical temperature and condensate fraction

If one defines  $T_c$  as the critical temperature at which BEC occurs, for any  $T < T_c$ , the ground-state is macroscopically filled. Since, at the critical temperature and in the thermodynamic limit,  $n_{E_0}(T_c) = 0$ , from Eqs. (6) and (11) one has that the equation allowing to determine  $T_c$  as a function of the filling  $f$  and of the hopping strength  $t$  reads as

$$\pi f = \int_{-2t}^{2t} \frac{dE}{\sqrt{4t^2 - E^2}} \frac{1}{e^{(E-E_0)/(k_B T_c)} - 1} \quad (12)$$

Equation (12) can be solved numerically for any value of  $f$ . When  $f \gg 1$ , one may expand the exponential in Eq. (12) to the first order in the inverse of the critical temperature  $T_c$  getting

$$\frac{\pi f}{k_B T_c} \approx \int_{-2t}^{2t} dE \frac{1}{\pi \sqrt{4t^2 - E^2}} \frac{1}{E - E_0}. \quad (13)$$

Substituting  $\cos \theta = E/2t$  in Eq. (13), one has

$$\frac{2t\pi f}{k_B T_c} = \int_0^\pi \frac{d\theta}{\cos \theta - E_0/2t},$$

from which



$$k_B T_c = E_J \sqrt{\left(\frac{E_0}{2t}\right)^2 - 1}, \quad (14)$$

with  $E_J$  the Josephson energy defined in Eq. (2). The result (14) holds for any graph for which  $E_0 < -2t$  and the density of states of the continuum part of the spectrum is given by Eq. (6), i.e., the density of states of a linear chain. Since, for a comb lattice  $E_0 = -2\sqrt{2}t$  [2], for this graph one gets  $k_B T_c = E_J$ . For a linear chain one has instead  $E_0 = -2t$  and thus  $T_c = 0$ : of course, no condensation occurs in this case.

Upon inserting the value of the ground-state energy (7) in Eq. (13), the critical temperature  $T_c$  is given by

$$k_B T_c \approx \frac{p-2}{2\sqrt{p-1}} E_J. \quad (15)$$

Equation (15) has been checked numerically and it is in excellent agreement with the numerical solution of Eq. (12): for  $f \gg 1$ , the error is of order  $1/f$ . For interwell barriers  $V_0$  of order  $\sim 15E_R \sim 2\pi\hbar \cdot 50kHz$  and for fillings  $f \sim 200$  one has  $E_J \sim 50nK$ . According to Eq. (15), one then expects the formation of an observable condensate in the star center. In Fig. 4 we plot the critical temperature  $T_c$  given by Eq. (15) as a function of the number of arms.

One may use Eq. (15) to determine also the condensate fraction as a function of the scaled temperature  $T/T_c$ . In the thermodynamic limit, the number of particles in the delocalized states is given by

$$N_{\sigma_0}(T/T_c) = \lim_{L \rightarrow \infty} N_s \int_{-2t}^{2t} \rho(E) \frac{dE}{e^{\beta(E-E_0)} - 1} \approx N_T \cdot \frac{T}{T_c}. \quad (16)$$

In Eq. (16) the exponential has been expanded to the first order in  $\beta$ : this approximation holds for  $f \gg 1$  and it is in very good agreement with the numerical evaluation of the integral (16) also in a large neighborhood below  $T_c$ . The critical temperature at which BEC occurs crucially depends on the number of arms of the star (see Eq. (15)) and thus one may adjust it by choosing a pertinent number of arms.

From Eqs. (11) and (16), one gets the number of particles in the localized ground-state  $N_{E_0}$ : the fraction of condensate, for  $T < T_c$ , is then given by

$$\frac{N_{E_0}}{N_T} \approx 1 - \frac{T}{T_c}. \quad (17)$$

For  $f$  ranging from  $10^3$  to  $10^9$ , the results provided by Eq. (17) differ from those obtained by the numerical evaluation of  $N_{E_0}$  from Eq. (11) by less than 1%. Equation (17) clearly shows that the condensate has dimension 1; cigar-shaped one-dimensional atomic Bose condensates support, in fact, a condensate fraction given by Eq. (17) [17,18].

### B. Distribution of bosons along the arms of the star network

In the following we shall determine the distribution of the bosons over the star graph. Due to the topology induced spatial condensation in the center of the star graph, one should expect an inhomogeneous distribution of the bosons along the arms of the network. The average number of bosons  $N_B$  at a site  $(x, y)$  depends - due to the symmetry of the graph - only on the distance  $x$  from the center of the star. At any temperature,  $N_B$  is given by:

$$N_B(x; T/T_c) = \lim_{L \rightarrow \infty} \left\{ N_{E_0}(x; T/T_c) |\psi_{E_0}(x)|^2 + (pL + 1) \int_{-2t}^{2t} dE \rho(E) \frac{1}{e^{\beta(E-E_0)} - 1} |\psi_E(x)|^2 \right\}. \quad (18)$$

In Eq. (18)  $\psi_{E_0}(x)$  is the wavefunction corresponding to the ground-state of the single-particle spectrum and  $\psi_E(x)$  is the wavefunction associated to a delocalized state with energy  $E$ . The last term in the right-hand side of Eq. (18) gives then the contribution coming from the delocalized states, which, in the thermodynamic limit, is independent from the site index  $x$  and equals the constant  $(N_T/N_s) \cdot (T/T_c)$ . Using Eq. (17), for  $T < T_c$ , one has

$$N_B(x; T/T_c) \approx \lim_{L \rightarrow \infty} N_T \left\{ \left(1 - \frac{T}{T_c}\right) \frac{p-2}{2p-2} e^{-x \log(p-1)} + \frac{1}{pL+1} \cdot \frac{T}{T_c} \right\}. \quad (19)$$

The exponential behavior of the ground-state eigenfunction leads, for  $T < T_c$ , to an increase of  $N_B$  on the sites near the center of the star while, when  $x \gg 1$ , the behavior is dominated by the last term in the right-hand side of Eq. (19). Thus, away from the center, once the filling is fixed,  $N_B$  depends only on the scaled temperature  $T/T_c$  and it is given by

$$\frac{N_B(x; T/T_c)}{f} \approx \frac{T}{T_c}. \quad (20)$$

Topology induced spatial BEC in a system of non-interacting bosons hopping on a star graph predicts then a rather sharp decrease of the number of bosons at sites located away from the center. The linear dependence exhibited by the solid line in Fig. 5 is consistent with the observation that, in this system, the condensate has dimension 1.

#### IV. CONCLUDING REMARKS

We showed how the topology of an optical lattice confining a dilute atomic gas may catalyze the existence of new and unexpected coherent phases. For this purpose we analyzed the paradigmatic and simple example of bosons hopping on star graph; our analysis allowed not only for the computation of the critical temperature  $T_c$  for which there is condensation of the bosons in the center of the star, but also allowed us to compute the distribution of the bosons along the arms of the star and to show the simple dependence of the non-condensate fraction on the reduced temperature  $T/T_c$ . We find  $T_c \propto tf$  where  $t$  is the tunneling rate and  $f$  the average number of particles per site.

In this paper we analyzed the behavior of bosons hopping on a star shaped network in the thermodynamic limit. It is comforting to observe that numerical simulations point out to the fact that - already for  $f \sim 100$  and for a reasonable number ( $L \sim 50$ ) of lattice sites on each arm of the star graph - finite size effects are negligible and, thus, the results derived in this paper are also very useful to pertinently describe the variety of experimentally accessible systems for which it is expected to observe the signature of a topology induced BEC. Although we focused our attention to bosons hopping on star shaped optical networks, the reader may easily convince her(him)-self that our analysis could be also applied to the description of topology induced coherent phenomena in star shaped Josephson junction networks [2]. In the latter application, there is practically no limitation on the number of junctions needed to build the star shaped network and, thus, the results obtained in the thermodynamic limit are expected to be very accurate.

An experimental realization of a star-shaped optical lattice with four (or six) arms may be achieved by first creating a regular square (or cubic) lattice using two pairs of counterpropagating laser beams; for these configurations, the optical potential has the form  $V(x, y) = V_0[\sin^2(kx) + \sin^2(ky)]$  (or similarly  $V(x, y, z) = V_0[\sin^2(kx) + \sin^2(ky) + \sin^2(kz)]$ ). A row and a column of the square lattice (or three perpendicular chain of the cubic lattice) may then be selected by superimposing two (or three) perpendicular gaussian laser beams, obtaining by this procedure a four- (six-) arm star optical lattice. Typical experimental values of  $V_0$  for which the tight-binding approximation and the Bose-Hubbard Hamiltonian (1) are valid are  $V_0 \sim 10 - 30E_R$  ( $E_R = \hbar^2/2m\lambda^2$  with  $\lambda$  the wavevector of the lattice beams, with  $\lambda \sim 800nm$ ). With an average number of particle per site  $f \sim 200$ , from Eq. (15) one finds (for  $V_0 \approx 15E_R$  and  $p = 6$ )  $T_c$  of the order of  $50nK$ . Below  $T_c$ , the macroscopic occupation of the ground-state could be evidenced by turning off the magnetic+optical trap and observing the gas expansion, as in the usual detection of atomic Bose-Einstein condensates.

The experimental observation of topology induced spatial BEC on a star shaped network may be easily achieved also using superconducting Josephson junctions. For a superconducting network, it is sufficient to measure the  $I$ - $V$  characteristic of a single arm of the Josephson junction network (JJN) built on a star graph and of the Josephson critical current along a given arm; if, in fact, one feeds an external current  $I_{ext}$  at the extremities of the arm, one expects to observe no voltage unless  $I_{ext}$  is larger than the smallest of the critical currents of the junctions along the arm. Since, below  $T_c$ , the Josephson critical current of the arm is given by the smallest of the critical currents of the junctions positioned along the arm, the measurement of the  $I$ - $V$  characteristic of an arm of the star graph should provide a measurement of the critical current of the junction which is farther from the star center. From the analysis carried out in this paper, it is rather easy to provide an estimate of the Josephson critical current as a function of both the temperature and the distance from the star center above and below  $T_c$ . Furthermore, one may show that the ratio of the Josephson critical currents of a Josephson junction - located on a given arm between the sites  $(x + 1, y)$  and  $(x, y)$  - above and below the critical temperature  $T_c$  does not depend on  $y$  and it is given by

$I_c^B(x, \tau)/I_c^A(x) \approx \sqrt{N_B(x+1; \tau)N_B(x; \tau)}/f$ , where  $I_c^B$  ( $I_c^A$ ) is the Josephson critical current below (above)  $T_c$ ,  $\tau \equiv T/T_c$  is the scaled temperature and  $N_B(x; \tau)$  is given by Eq. (20). Far away from the star center ( $x \gg 1$ ), one gets

$$\frac{I_c^B(x, \tau)}{I_c^A(x)} \approx \frac{T}{T_c}. \quad (21)$$

Thus, BEC in a star shaped JJN predicts a sharp decrease of the Josephson critical current for a junction located away from the center.

The striking and intriguing similarities between superconducting Josephson junction networks and atomic gas in suitable deep optical lattices have been already pointed out [6]: one may think, in fact, to realize a star shaped network also using bosonic Josephson junctions. For this purpose, it is needed that, in each well of the periodic potential, there is a condensate grain appearing at a Bose-Einstein condensation temperature  $T_{BEC}$  and that, when  $T_{BEC}$  is larger than all other energy scales, the atoms in the  $i$ -th well of the optical lattice may be described by a macroscopic wavefunction  $\psi_i$ . Thus, it becomes apparent that an optical network may be regarded as a network of bosonic Josephson junctions. Our analysis shows that - at a temperature  $T_c < T_{BEC}$  - the topology of the star-shaped network induces a further finite temperature transition to a state in which the bosons spatially condense in the center of the star.

## ACKNOWLEDGEMENTS

Discussions with M. Rasetti and A. Smerzi are gratefully acknowledged. We acknowledge financial support by M.I.U.R. through grant No. 2001028294.

## APPENDIX A: ENERGY SPECTRUM ON A STAR-SHAPED NETWORK

In this Appendix we solve the eigenvalue equation (4) with the adjacency matrix given by Eq. (5), describing bosons hopping on a star graph with  $(pL + 1)$  sites (i.e., a star with  $p$  arms having  $L$  sites each).

Since non-interacting bosons on a linear chain are described by plane waves with wave vector  $k$  one has that, on each arm, an eigenstate of Eq. (4), corresponding to energy  $E = -2t \cos(k)$ , may be written as:

$$\psi(x, y) = A_y e^{ikx} + B_y e^{-ikx}. \quad (\text{A1})$$

In Eq. (A1)  $y = 1, \dots, p$  is an index labeling the arm, while  $x = 0, \dots, L$  labels the sites on the arm. The wavefunctions described in Eq. (A1) are, of course, delocalized.

Requiring that - on each arm -  $\psi(x, y)$  is a solution of the eigenvalue equation at  $x = L$  amounts to require that  $A_y$ ,  $B_y$  and  $k$  should satisfy the  $p$  equations

$$-t \left( A_y e^{ik(L-1)} + B_y e^{-ik(L-1)} \right) = -2t \cos(k) \left( A_y e^{ikL} + B_y e^{-ikL} \right) \quad y = 1, \dots, p. \quad (\text{A2})$$

Furthermore, the eigenstates defined in Eq. (A1) should satisfy  $(p-1)$  matching conditions in the center of the star where the wavefunctions defined on each arm are linked; thus, one has

$$A_y + B_y = A_{y+1} + B_{y+1} \quad y = 1, \dots, p-1. \quad (\text{A3})$$

Using Eqs.(A3), the condition in the center gives one more equation

$$-t \sum_{y=1}^p \left( A_y e^{ik} + B_y e^{-ik} \right) = -2t \cos(k) (A_{y'} + B_{y'}) \quad y' = 1, \dots, p. \quad (\text{A4})$$

Equations (A2) and (A4) may be grouped in a homogeneous linear system of  $2p$  equations which allows to fix the  $2p$  parameters  $A_y$  and  $B_y$ . Upon denoting with  $M$  the  $(2p \times 2p)$  matrix whose elements are the coefficients of the linear system given by Eqs. (A2) and (A4), requiring that

$$\det M = \Theta(k, L) \cdot (1 - e^{2ik(L+1)})^{p-1} \cdot \{(p-2) \cot(k) - p \cot[k(L+1)]\} = 0, \quad (\text{A5})$$

guarantees the uniqueness of the solution. In Eq. (A5)  $|\Theta(k, L)| = 1$  for any value of  $k$ .

One immediately sees that the values of  $k$  for which  $k = n\pi/(L+1)$  (with  $n = 1, 2, \dots, L$ ) provide a set of  $L$   $(p-1)$ -fold degenerate eigenstates of Eq. (A5). In addition, the solutions of the transcendental equation

$$(p - 2) \cot(k) - p \cot[k(L + 1)] = 0 \quad (\text{A6})$$

provide the values of  $k$  associated to non-degenerate eigenstates. Equation (A6) can be solved numerically and yields a set of  $(L - 1)$  non-degenerate eigenvalues corresponding to values of  $k$  which - in the thermodynamic limit - are equally spaced and separated by a distance  $\pi/(L + 1)$ . As a result, the set of delocalized states is formed by  $(pL - 1)$  states corresponding to energies ranging between  $-2t$  and  $+2t$ . One can easily convince oneself [3] that, in the thermodynamic limit, the normalized density of states is given by

$$\rho(E) = \frac{1}{\pi \sqrt{4t^2 - E^2}}, \quad (\text{A7})$$

as in the case of non-interacting bosons hopping on a linear chain (see e.g. [13]).

Since the total number of states should equal  $N_s = (pL + 1)$ , there are also two localized states in the spectrum: to find them, it is convenient to look for solutions of the eigenvalue equation (4) of the form

$$\begin{aligned} \psi_0(x) &= Ae^{-\eta x} + Be^{\eta x} \\ \psi_+(x) &= A(-1)^x e^{-\eta x} + B(-1)^x e^{\eta x} \end{aligned} \quad (\text{A8})$$

corresponding, respectively, to the eigenvalues  $E_{0,+} = \mp 2t \cosh \eta$ . In Eqs. (A8)  $A$  and  $B$  are normalization constants and  $\eta \equiv 1/\xi$  is a parameter accounting for the localization of the states. One may determine the parameters  $A$ ,  $B$  and  $\eta$  by using only the normalization condition for  $\psi_0$  and by rewriting Eq. (A6) for  $k = i\eta$ . Namely, one should solve:

$$(p - 2) \coth(\eta) - p \coth[\eta(L + 1)] = 0, \quad (\text{A9})$$

together with the condition that  $\sum_{x,y} |\psi_0(x, y)|^2 = 1$ . For  $L \rightarrow \infty$ , one can always set  $B = 0$ . Equation (A9) becomes  $(p - 2) \coth(\eta) - p = 0$  which is solved by  $\eta = \log(p - 1)/2$  yielding  $E_0 = -tp/\sqrt{p - 1}$  and  $E_+ = -E_0$ . Solving the eigenvalue equation (4) for  $E = E_{0,+}$ , one obtains the wavefunctions of both the localized states. The normalized eigenfunction for the ground-state is then given by:

$$\psi_{E_0}(x) = \sqrt{\frac{p-2}{2p-2}} e^{-x/\xi}. \quad (\text{A10})$$

Since, for  $L \rightarrow \infty$ , the normalized density of the continuum states is given by  $\rho(E) = 1/(\pi\sqrt{4t^2 - E^2})$ , the two localized states do not contribute to the closure relation in the thermodynamic limit; thus, they belong to the hidden spectrum.



# FIGURES

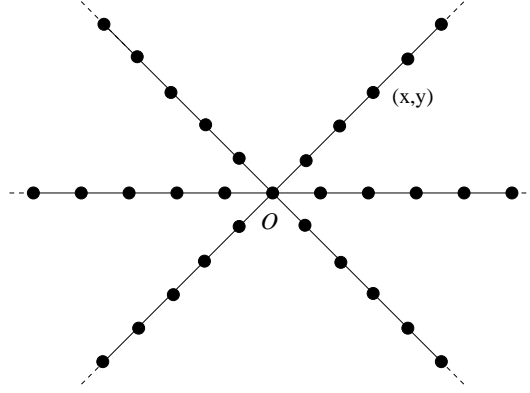


FIG. 1. A star network with six arms.

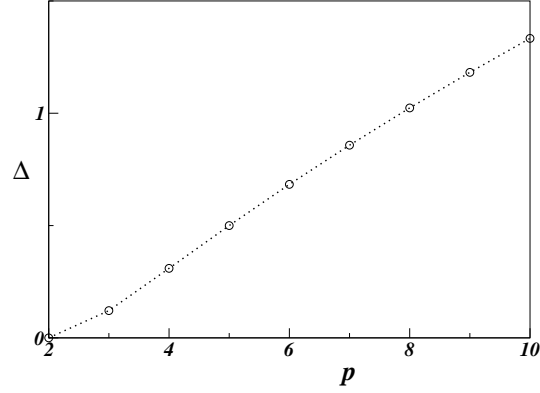


FIG. 2. Energy gap  $\Delta$  between the ground-state energy and the continuum part of the spectrum (in units of the tunneling rate  $t$ ) as a function of the number of arms  $p$ . For  $p = 2$  (i.e., a linear chain)  $\Delta = 0$ . The dotted line is just a guide to the eye to connect the points.

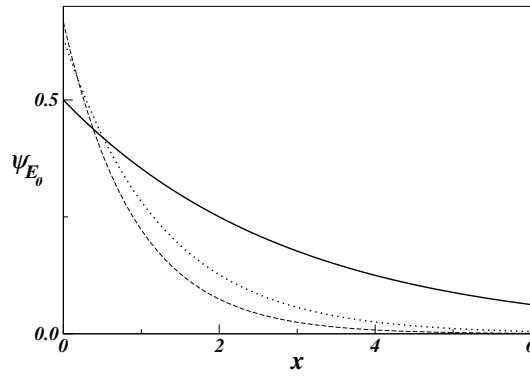


FIG. 3. The normalized single-particle ground-state wavefunction for bosons hopping on a star graph as a function of the distance  $x$  from the center. The number of arms  $p$  is respectively 3 (solid line), 6 (dotted line), and 10 (dashed line).

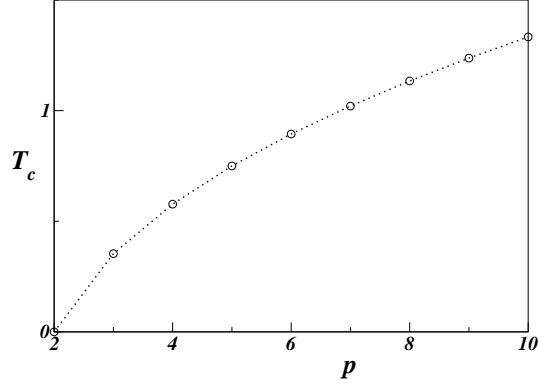


FIG. 4. Critical temperature  $T_c$  (in units of  $k_B/E_J$ ) as a function of the number of arms  $p$ . For the linear chain ( $p = 2$ )  $T_c = 0$ .

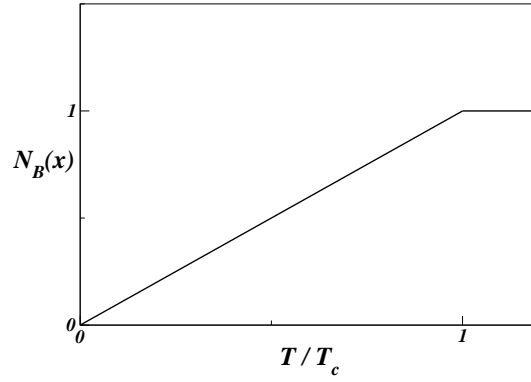


FIG. 5. Distribution of the number of bosons  $N_B$  as a function of  $T/T_c$  computed for  $x \gg 1$ .  $N_B(x)$  is in units of the filling  $f$  and is therefore equal to 1 for  $T \geq T_c$ .

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